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# BRIEF REPORTS

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## Self-organized criticality can emerge even if the range of interactions is infinite

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We have investigated a model of tree growth in order to show that self-organized criticality (SOC) can emerge even if perturbations due to long-range interactions are propagating throughout the whole system. The interaction potential is assumed to be a power $d^{-\alpha}$ of the distance $d$ measured along the tree between branch extremities. A transition occurs for $\alpha_c = 1$ according to simulation results between SOC ($\alpha > \alpha_c$) and non-SOC ($\alpha < \alpha_c$) regimes. A theoretical treatment supports the idea that the transition occurs when the exponent $\alpha$ is equivalent to the fractal dimension $D_f^S$ of the backbone (skeleton) of the tree. [S1063-651X(98)09101-6]

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### I. INTRODUCTION

Self-organized criticality (SOC) has received much attention since it was proposed to be a paradigm for the description of a wide variety of dynamical processes [1]. The SOC behavior is the tendency for large systems composed of many interacting agents to evolve spontaneously towards a critical state. In the latter, the system exhibits long-range power-law correlations in space and time while the interactions between agents usually take place over a finite range like nearest-neighboring columns of sand grains in the Bak-Tang-Wiesenfeld (BTW) model.[1–3] The SOC behavior is in contrast with classical phase transitions for which the fine tuning of an external field (a temperature-like parameter) is needed for a critical situation. Moreover, a critical state with self-organization is characterized by an intermittence of ‘‘avalanches’’ [4]. Much experimental and numerical evidence of SOC has been found: cellular automata, sandpiles, earthquakes, vortex motion in type-II high-$T_c$ superconductors, invasion percolation, fractal growth, and interface pinning [4].

Recently, we have extended the Bak-Sneppen (BS) model [5] of SOC to the growth of trees through successive branching events [6]. A random scalar number $f_i$ between zero and one is associated with each leaf of a tree. Physically, this number is like a bound state in a potential or the degree of stability of the extremity. At each growth step, a branching event is supposed to take place on the leaf having the lowest $f_i$ value, i.e., the least stable. The model is thus driven by so-called *extremal dynamics*. The branching event leads to two offsprings, each receiving a new random potential value. Interactions are introduced in the system by assuming that each branching event affects the $f_i$'s of other leaves when these are separated from the branching point by a distance $d_{ij}$ less than a parameter $k$ having a finite value. The distance $d_{ij}$ between $i$ and $j$ is defined as the minimum number of segments needed to connect both leaves. The above selection-branching perturbation process is repeated a desired number $t$ of times.

Each self-organized critical state is characterized by a set of *critical exponents* derived from a power-law behavior of relevant properties as a function of scale, time, or internal parameter $f - f_c$. It has been demonstrated [7] that the finite $k$-value tree model does not belong to the SOC universality class of extremal dynamics models such as invasion percolation, BS, sandpiles, and BTW. Indeed, the relationships between critical exponents that have been defined [8] for extremal dynamics models do not hold for SOC tree models [7]. This arises from the fact that the SOC tree model generates its own fractal structure of dimension $D_f$ in which perturbations are propagating along the backbone (skeleton) of dimension $D_f^S = D_f - 1$ [9], in contrast to the majority of SOC models which are defined on a regular and static lattice. Another original feature of the SOC tree model is that the values of the critical exponents depend on the range $k$ of the interactions between agents (extremities) [6].

It is natural to expect some SOC behavior when the agents have a finite range of ‘‘interactions’’ between them. This is certainly true in the Bak-Sneppen co-evolution lattice model since, in this case, some regularity is introduced into the evolution process and only a limited amount of elements are changing at each step. In the case of an infinite range of interaction, SOC is expected to disappear if there is no probability condition for the way an agent is changing state. This is the case of the random-neighbor version of the Bak-Sneppen model which is recognized to be noncritical [17]. The destruction of SOC in the case of random-neighbor sandpiles has also been reported [3].

For the SOC tree models, when $k \to \infty$, all agents are in-

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*Author to whom correspondence should be addressed. Electronic address: vandewal@gw.unipc.ulg.ac.be teracting. Then all $f_i$'s are modified at each time step and the SOC behavior is expected to be completely destroyed [16]. No pattern should be expected, since there is no memory effect at all, and all time steps look like the first one. If there is no order at first, there will never be any order later on. For a tree, when $k \rightarrow \infty$, the tree growth reduces to a simple Eden growth [10] as with a Cayley tree. Notice that such a situation is not artificial since very long-range interactions as well as a treelike (hierarchical structure) growth are relevant features for turbulent economic systems, where evidence of power laws as in SOC have been reported [11]. Long-range interactions are also relevant for the Internet growth for which $1/f$ noise has been recently reported as well [12].

Thus, it is of fundamental, theoretical, and practical interest to investigate whether or not there are conditions for conserved SOC behavior *even if all agents are interacting.* In order to do so, we have investigated the case of a tree (SOC) when $k \rightarrow \infty$, i.e., when all agents are interacting. Ingredients for the emergence of SOC are found, though for a somewhat surprisingly simple condition.

## II. MODEL

The present study considers the tree model with $k=\infty$ and with leaf interactions given by a power law of the distance to the branching node. When a branching event occurs on a leaf $i$ having the minimum potential value $f_i$, the process leads to two new offspring. The ''potential'' of each offspring $m$ is supposed to be correlated to the value of the parent by the law

$$f_m \leftarrow f_i + \epsilon_m , \tag{1}$$

where $\epsilon_m$ is a random number in the interval $[f_i - 1, 1 - f_i]$. Moreover, due to the infinite range of the interactions, the potential $f_j$ for all leaves $j$ of the tree is assumed to be updated, following

$$f_j \leftarrow f_j + \frac{\epsilon_j}{d_{ij}{}^{\alpha}}, \tag{2}$$

where $\epsilon_j$ is a random number belonging to the interval $[f_j - 1, 1 - f_j]$. The $f_j$ values are constrained to remain in the interval $[0,1]$. The exponent $\alpha$ is the key parameter that is going to control the propagation of the potential landscape perturbations along the tree. For $\alpha = 0$, one recovers the tree-SOC model with $k=\infty$ and for which the SOC behavior is destroyed. For $\alpha \rightarrow \infty$, the model reduces to the invasion percolation rule of a Cayley tree that exhibits SOC.

The Fisher algorithm [14], for which each branching node is labeled with reference to the label of its ancestor, has been used in order to speed up simulations. The CPU computing time increases as a power of $t$ with an exponent close to 3. This value is in contrast to an exponent close to 2 for the previous tree-SOC model [7]. Within a reasonable computation time, it has been difficult to grow trees with more than $10^4$ branching points. Nevertheless, our results of ''small'' trees will be seen to be relevant for the present quest of SOC behavior.

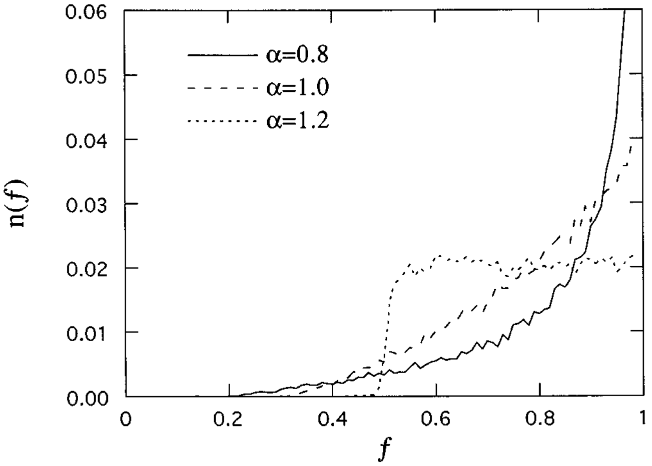

FIG. 1. Distribution $n(f)$ of $f_i$'s values for three different values of $\alpha$: 0.8, 1.0, and 1.2.

## III. NUMERICAL RESULTS

Starting from a single root with any value of the initial potential, the distribution $n(f)$ of potential values for the leaves is seen to self-organize always towards a steady state. Typical distributions $n(f)$ of the $f_i$'s after $t=10^4$ growth steps are shown in Fig. 1. Three different situations are illustrated: $\alpha=0.8$, $\alpha=1.0$, and $\alpha=1.2$. The shape of the distribution $n(f)$ is seen to change drastically at $\alpha_c=1$. Below $\alpha_c$, the distribution $n(f)$ seems to be a continuous and increasing function between 0 and 1. Above $\alpha_c$, $n(f)$ is a steplike distribution with the great majority of the $f_i$'s above a gap or threshold $f_c$. This means that all branching events have taken place through leaves having $f_i < f_c$. The threshold is close to $f_c \approx 0.5$ for $\alpha > \alpha_c$. The formation of such a gap $f_c$ in the $n(f)$ distribution for $\alpha > \alpha_c$ leads naturally to the definition of avalanches [5] as a causally connected sequence of activity below $f_c$. It should be noted that the threshold is of course not well defined numerically for $\alpha \approx \alpha_c$ due to the necessary finiteness of the simulated trees. For $\alpha$ close to $\alpha_c$, it has thus been difficult to measure precisely the distribution of avalanches in order to demonstrate a hypothetical SOC (or change in) behavior. However, evidence of avalanche dynamics as well as criticality can be given about other features of the tree growth as seen below.

Figure 2 presents the two-dimensional projection of two trees for $\alpha=0.5$ and $\alpha=2.0$, respectively. These trees are made up of 4000 branching nodes; each one is represented by a dot. The vertical axis represents the distance from the root while the horizontal axis is arbitrary. The grey levels represent different steps of the growth. The skeleton of each tree is for $\alpha < \alpha_c$ and the trees are found to be dense objects close to Cayley trees. Above $\alpha_c$, the trees are fractal objects, thus reflecting a SOC behavior driving the tree growth itself, as in the case of finite range of interactions [6,7]. For $\alpha > \alpha_c$, the fractal dimension of the trees has been found to be close to $D_f=2$, a result that is different from the tree-SOC model of Ref. [6] for which, e.g., $D_f=1.92 \pm 0.04$ for $k=2$ [7].

Figure 3 presents a semilog plot of the evolution of the largest distance $d_{\max}$ from the leaf top to the root of the tree. The $\alpha=0.5$ and $\alpha=2.0$ cases are illustrated. Both cases can

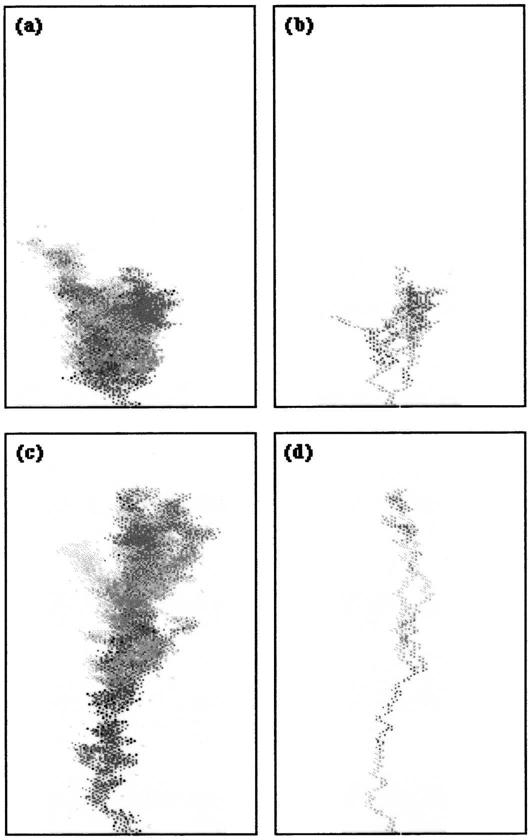

FIG. 2. The two-dimensional projection of two trees for (a) $\alpha=0.5$ and (c) $\alpha=2.0$, respectively. These trees are made up of 4000 branching nodes, each represented by a dot. The grey levels represent different steps of the growth. The vertical axis represents a measure of the distance from the root while the horizontal axis has an arbitrary scale. The skeletons (b) and (d), i.e., the backbone, of both trees (a) and (c) are also illustrated.

be described by successive jumps of $d_{\max}$ and periods of stasis. The jumps seen in the $d_{\max}$ time evolution are avalanches at the ''top'' of the tree, while during the periods of stasis the activity takes place in the ''inner parts'' of the tree.

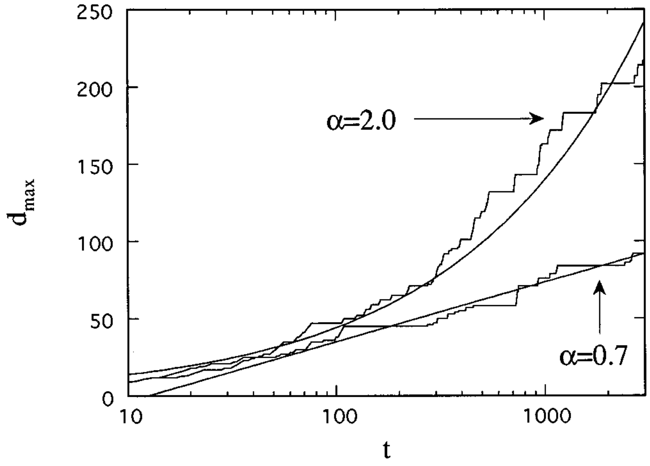

FIG. 3. Log-log plot of the time evolution of the length $d_{\max}$ of the tree. Two different values of $\alpha$ are illustrated for conciseness: 0.5 and 2.0.

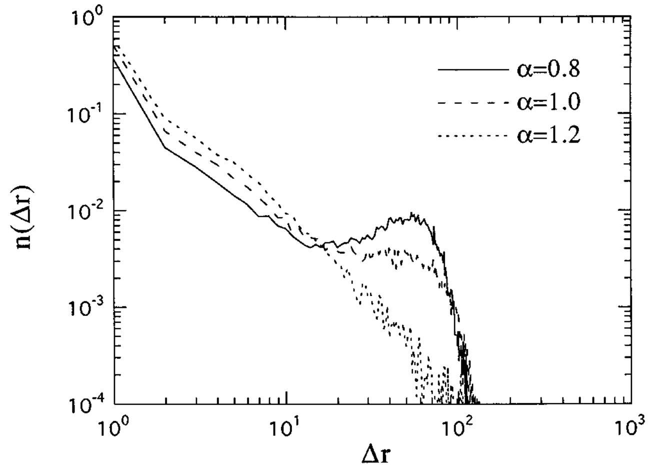

FIG. 4. Log-log plot of the distribution $n(r)$ of distances $\tau$ measured along the tree between successive growth events. Three different values of $\alpha$ are illustrated: 0.8, 1.0, and 1.2.

For $\alpha=0.5$, $d_{\max}$ has clearly a logarithmic evolution expressing the nonfractality of the tree for $\alpha<\alpha_c$. For $\alpha=2.0$, $d_{\max}$ increases as a power of $t$ with an exponent $\frac{1}{2}$, expressing that the tree is fractal with an exponent $D_f=2$. Thus, the investigation of the tree geometry clearly supports the idea of a transition between non-SOC and SOC regimes at $\alpha_c$.

We have also investigated the jumps of the branching activity on the tree, i.e., the (shortest) distance $r$ measured along the tree between two consecutive growth events. Figure 4 presents in a log-log plot the distribution $n(r)$ between successive branching leaves. Three different situations are illustrated: $\alpha=0.8$, $\alpha=1.0$, and $\alpha=1.2$. Below $\alpha_c$, the distribution of jumps presents a cut off and is smoothly curved in a log-log plot. Moreover, a ''bump'' appears near the cut off. This bump comes from the fact that the distance between two successive growth events has roughly the size of the tree for a random (Eden-like) tree growth. This bump disappears for $\alpha>\alpha_c$. Above $\alpha_c=1$, the $n(r)$ distribution is a power law $n(r)\sim r^{-\pi}$ with an exponent $\pi=1.9\pm0.1$. Such a power-law behavior is a characteristic of SOC processes [8].

## IV. DISCUSSION

Let us consider theoretical arguments next. It has been recalled that when $\alpha\rightarrow\infty$, the model reduces to the invasion percolation problem of a Cayley tree or to a critical branching process [13] for which $f_c=\frac{1}{2}$, $D_f=2$, and $\pi=2$ are exact values. When $\alpha$ is finite, a mean-field treatment of the tree-growth problem is possible by distinguishing *active* and *inactive* leaves. Let us define an *active* leaf $i$ of the tree as one with $f_i<f_c$, while an *inactive* one is for $f_i>f_c$. In a mean-field approximation, the modulation of the perturbation results from an average over all leaves such that the evolution equations read

$$N_{\rm act}(t+1)=2\,f_c N_{\rm act}(t)+p(f_c,\langle d^{\alpha}\rangle)N_{\rm inact}(t), \tag{3}$$

and

$$N_{\mathrm{inact}}(t+1)=2(1-f_c)N_{\mathrm{act}}(t)+[1-p(f_c,\langle d^{\alpha}\rangle)]N_{\mathrm{inact}}(t), \tag{4}$$

where $p(f_c,\langle d^{\alpha}\rangle)$ is the probability that an inactive leaf becomes active at a distance $\langle d^{\alpha}\rangle$ from the latest branching event. Assuming that $p$ has a simple form $p_1/\langle d^{\alpha}\rangle$, one finds that the number of active leaves $N_{\mathrm{act}}$ presents a stationary solution for $p_1/\langle d^{\alpha}\rangle\rightarrow 0$. In this case, the gap exists and $f_c=\frac{1}{2}$. Since the probability $n(d)$ of finding two points separated from a length $d$ is proportional to the power $d^{-D_f}$ where $D_f$ is the dimension of the embedding space (the tree herein), one has

$$\langle d^{\alpha}\rangle=\int_0^{t^{1/D_f}} n(d)\,\delta d = t^{(\alpha-D_f+1)/D_f}. \tag{5}$$

Thus, the formation of a gap $f_c$ is only possible for $(\alpha-D_f+1)>0$ such that a transition takes place at

$$\alpha_c=D_f-1. \tag{6}$$

This mean-field result is in quite good agreement with the numerical results presented above ($D_f\approx 2$ and $\alpha_c\approx 1$). It is conjectured that the relationship should apply to fractal trees grown under other branching conditions [9].

It should be pointed out that this critical exponent $D_f-1$ is equivalent to the fractal dimension $D_f^S$ of the tree skeleton [9], i.e., the backbone of the tree, for $D_f\geqslant 2$. This gives a strong physical meaning to the above result [Eq. (6)], since the perturbations are clearly mediating through the tree skeleton. The fact that the tree skeleton may play an important role during the growth is also related to the screening of extremities leading to fractal trees as discussed in Ref. [15].

## V. CONCLUSION

In summary, we have attempted to solve the dilemma concerning SOC conditions when the interaction range is infinite. We have investigated a simple model of tree growth for which the range of interactions between agents behaves like a power law $d^{-\alpha}$ of the distance separating the interacting agents. We have found that a transition between non-SOC and SOC growth regimes occurs at a critical value $\alpha_c=1$. This value is found to be the fractal dimension of the tree skeleton $D_f^S=D_f-1$, i.e., the dimension of the structure trough in which interactions are mediated.

*Note added in proof.* An interesting paper to consider with respect to the above work is that of Cannas [18], where it is shown that the critical properties of one-dimensional Ising models with long range interactions that decay like $1/r^{\alpha}$ depend much on $\alpha$, in particular for $1<\alpha\leqslant 2$.

## ACKNOWLEDGMENTS

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